## Riemann Integral

**H**istorically the concept of integration came into existence as a means of evaluating area under a curve, i.e. in compliance with a geometrical need. The first rigorous approach quite naturally therefore started based on intuitive ideas of sum and in effect as the limit of a sum, now-a-days known as Riemann sum. But when the limitation of this approach was exposed through different situations, a rigorous arithmetic approach was contemplated by G.F.B. Riemann (1826-1866) with remarkable success. This approach is now known as Riemann’s theory of integration and plays a fundamental role in analysis.

**SOME DEFINITIONS**

Let {*a, b*}, *a > b* be a closed interval of **R**.

***Definition* :** A partition of [a, b] is defined as a finite set of points {*x0, x1, ..... xr-1, xr,….., xn*} where a = *x0 x1 x2 …….xr-1 xr……… xn-1 xn*= b and will be denoted by *P*.

This gives us a finite collection of non-overlapping closed intervals [*xr-1 , xr*] (*r* = 1,2,……,n) whose union is [a, b]

The set of all partitions will be usually denoted by P [a, b]or simply P, if there is no chance of confusion with regard to the interval.

The subintervals associated with the partition *P* = { *x0, x1, ..... xr-1, xr,….., xn* } will be usually denoted as *I1* = [*x0, x1*], *I2* = [*x1 , x2*]……*In* = [*xn-1 , xn*]. The length of the interval *Ir* = [*xr-1 , xr*] wil be denoted by r .

Thus r = *xr* – 1. Clearly = *b-a*.

***Definition*:** A partition *P\** is called a *refinement* of the partition *P* if *P P\** .

**Observation 1:** Clearly, if *P1, P2 P* [a, b] then *P1 P2*  is a *refinement* of *P1* and also of *P2.*

**Observation 2:** For every positive number we can find a partition **P** *P* [a, b] with

**Observation 3:** If *P\** is a refinement of *P* , then

**Observation 4:** If *P\*\** is a refinement of *P\** and *P\** is a refinement of [a, b], then *P\*\** is a refinement of *P.*

**Observation 5:** If *P*  = {*x0, x1, x2, ….. ,xn*} is a partition of [a, b] then *n* → ∞ does not mean ∞→ 0.

**Example 1:** The set *P* = {0, 1/4, 3/4, 1} is a partition of [0, 1]. The set *P\** = {1/4, 1/2, 3/4} is a refinement of *P*, but *P’* = {0, 1/3, 2/3, 1} is not a refinement of *P*.

Note, = ½ , = ¼ , = .

***Definition*  :** Let ƒ : [a, b]→**R** be bounded and *M*  and *m* be the upper and lower bounds of ƒ on [a, b] respectively. Now, if *Mr*  and *mr*  denote the upper and lower bounds of ƒ on *Ir* = [*xr-1, xr*], then clearly

*m*  *mr MrM* for r = 1,2, ……,n

The sums *U* (*P, ƒ*) = rr  and *L*(*P, ƒ*) = rr are called respectively the upper and lower sums of ƒ for the partition of *P*. The sum

*W (P, ƒ)* = *U* (*P, ƒ*) *L* (*P, ƒ*) = r *mr*)r

is called the *oscillatory sum*  of ƒ for the partition P.

The numbers inf *U* (*P, ƒ*) and sup *L* (*P, ƒ*) are called respectively the *upper*  and *lower integrals* of ƒover [a, b] and are denoted by

Thus  
  
   
**Observation 1:** *m* (b-a)≤ *L*(*P, ƒ*)≤ *U*(*P, ƒ*)≤ *M* (b-a).

**Proof :** We know

*m* ≤ *mr* ≤ *Mr* ≤ *M.*

Multiplying by r both sides and then taking summation, we get

Or *m* (b-a) ≤ *L* (*P, ƒ*)≤ *U* (*P, ƒ*)≤ *M* (b-a).

**Observation 2:** For every there exits *P* *P* [a, b] such that

Similarly for every , there exists *P’* such that

**Proof :** Follows from the definition of *U* (*P, ƒ*) and *L* (*P, ƒ*) as *infimum* and *supremum.*

**Observation 3:** If *P\** is a refinement of *P* and ≤ *k* for all *x* then,

1. *L*(*P, ƒ*) ≤ *L*(*P\*, ƒ*) ≤ *L*(*P, ƒ*) 2*pk*and *L*(*P\*, ƒ*) *L*(*P, ƒ*) ≤ (*M-m*)*p*.
2. *U*(*p, ƒ*) ≥ U(*P\*, ƒ*) ≥ *U*(*P, ƒ*) 2*pk*and *U* (*P, ƒ*) *U*(*P\*, ƒ*) ≤ (*M, m*)*p*.

Where = and *P\** has *p* additional points than *P.*

**Proof :** We first prove the result when *P\** contains only one additional point say, in *Ir* = [*xr-1, xr*]. Let *m’r* and *m”r* denote the lower bounds of ƒ on [*xr-1,* ] and [r] respectively and let *M’r* and *M”r,* denote the upper bounds of ƒ on [*xr-1,* ] and [r] respectively.

Then, clearly *mr* ≤ *m’r* , *mr* ≤ *m”r* , *M’r* ≤ *Mr , M”r* ≤ *Mr* .

*mrr* = *m* (*xr xr-1*) = *mr* (*xr* ) *mr* (r-1) ≤ *m’r*  (*xr* ) *m”r* ( *xr-1*)

*L* (*P\*, ƒ*) *L*(*P, ƒ*) = *mr*  (*xr* ) *m”r* (*xr-1*) *mr* (*xr r-1*)

= *m’r* (*xr* )*m”r* (r-1) *mr* (*xr* r-1)

= (*m’r* *mr*)(*xr* ) (*m”r r*)(*x xr-1*)

≥ 0 since *m’r* *mr*  ≥ 0, *m”r r* ≥ 0, *xr* ≥ 0, r-1 ≥ 0

Hence *L*(*P, ƒ*) ≤ *L* (*P\*, ƒ*)

We see also that

*L* (*P\*, ƒ*) *L* (*P, ƒ*) = *m’r*  (*xr*) *m”r* ( *xr-1*) *mr* (*xr* r-1)

= (*m’r r*)(*xr* ) (*m”rmr*)(r-1)

≤ 2*k*(*xr* ) 2*k*( *xr-1*) = 2*k*(*xr r-1*) = *2k*

since *m’r*  *mr* ≤ 2k and *m”r*  *m1* ≤ 2*k*.

Hence *L* (*P\*, ƒ*) ≤ *L* (*P, ƒ*) *2k*.

Further, *L* (*P\*, ƒ*) *L* (*P, ƒ*) = (*m’r mr*)(*xr* ) (*m”r mr*)(*xxr-1*).

≤ (*M* )(*xr* ) (*M m*)(*x xr-1*)

= (*M* )(*x xr-1*)≤ (*M* )

Now if instead of just one, *P\** contains *p*  additional points than those of *P*, for each such point we get the term *2k* and therefore for *p* points we get

*L* (*P\*, ƒ*) ≤ (*P, ƒ*) 2*kp and L* (*P\*, ƒ*) *L* (*P, ƒ*) ≤ (*M* )*p.*

*(ii* ) Similar to (*i)*.

**Observation 4:** If *P\** is a refinement of *P*, then

w(*P\*, ƒ*) ≤ w(*P, ƒ*)

**Proof :** By Observation 3, we have

*U* (*P\*, ƒ* ) ≤ *U* (*P, ƒ*)

*L* (*P\*, ƒ*) ≥ *L* (*P, ƒ*)

Hence *U* (*P\*, ƒ*) *L* (*P\*, ƒ*) ≤ *U* (P, ƒ) *L* (*P, ƒ*)

**Observation 5:** For *P1, P2* **P** [a, b]

*L* (*P1, ƒ*) ≤ *U* (*P2, ƒ*)

**Proof :** Let *P* = *P1  P2 ,* then *P*  is a refinement of *P1* and *P2* as well. Hence

*L* (*P1,* ƒ) ≤ *L*(*P, ƒ*) ≤ *U* (*P, ƒ*) ≤ *U* (*P2, ƒ*)

**Observation 6:** If ƒ : [a, b], then

**Proof :**  If *P1 , P2* **P** [a, b], then

Keeping *P2* fixed and varying *P1* over **P** [a, b], we get

*L* (*P1, ƒ*) ≤ sup *L* (*P, ƒ*) ≤ *U* (*P2, ƒ*)

i.e.

Now varying *P2* over **P** [a, b], we get

Hence

***Darboux’s Theorem:*** If ƒ : [a, b] → **R** is bounded, then to every > 0, there exists > 0 such that

and

for every *P* ***P*** [a, b] with < .

**Proof :** We prove only one of the inequalities. The proof of the other is similar. Since ƒ is bounded, there exists *k* **R** such that ≤ *k* for all *x* [a, b]. By Observation 2, to every there exists *P1* **P** such that

Let *P1* contains altogether *p* + 2 points which include the end points *a* and *b.*

Let > 0 be so chosen that = 2*kp i.e.* =

Now, if *P* be any partition with <, then we shall show that *P* **P** [a, b],

To this end, let *P2* = *P1* , i.e. *P2* is a refinement of *P1*  and *P*.

Then by Observation 4 , we get

*U* (*P, ƒ*) ≤ *U* (*P2 , ƒ*) 2*kp = U* (*P2, ƒ*)

Since *P2* is a refinement of *P1*, *U* (*P2, ƒ*) ≤ *U* (*P, ƒ*)

Hence

**1.2. RIEMANN INTEGRABILITY**

We now give the definition of Riemann integration and study some necessary and sufficient conditions for Riemann integrability of a function.

***Definition:***  A function ƒ : [a, b]→ **R** is said to be *Riemann integrable*  over [a, b] if

In this case the Riemann integral of ƒ over [a, b] is denoted by or Further (a<b) is denoted as provided ƒ is integrable over [a, b].

We shall denote the set of all Riemann integrable functions on [a, b] by *R* [a, b].

Thus ƒ **R** [a, b] implies (*i*) ƒ is bounded on [a, b] and (*ii*)

*Example 1 :* If ƒ(*x*) = *k* for all *x* [a, b], and *k* then ƒ[a, b].

*Proof :* Let *P* **P** [a, b]. Then

*U* (*P, ƒ*) = *k* (b-a), *L* (*P, ƒ*) = *k* (b-a)

Hence

and

Clearly and ƒ is bounded. Hence the result .

*Example 2:* The function ƒ defined on [a, b] as

ƒ(*x*) =

is not Riemann Integrable.

*Solution:* Let *P* [a, b]. Then clearly *M* = *Mr* = 1 and *m* = *mr* = 0 for every *r*. Hence *U* (*P,* ) = b-a and *L* (*P, ƒ*) = 0.

Therefore,

Hence *R* [a, b].

***Theroem 1.2.1. :*** (*a*) A function ƒ : [a, b]→ **R** is Riemann integrable if and only if for every there exists *P*  such that *w* (*P, ƒ*) < .

(b) A function ƒ : [a, b]→ **R** is Riemann integrable if and only if for every , there exists such that for *P* **P** [a, b] with <.

*w*(*P, ƒ*)<

*Proof :*  (a) Let ƒ *R* [a, b]

Then

Let

Then by Observation 2, there exists *P*  such that

Hence

Or

Conversely, let, for arbitrary >0, there exists *P*  **P** [a, b] such that *w* (*P, ƒ*) <

Then  
  
 But since  
  
 We get,

But is arbitrary . Therefore

Hence ƒ *R* [a, b] .

(b) Follows similarly if Darboux’s theorem is applied in place of Observation 2.

Before we take up the integrability conditions, we make two more observations.

**Observation 7:** *m* (b-a)≤ where *m* and *M*  are the bounds of ƒ over [a, b], a<b and ƒ is Riemann integrable.

**Proof :** If *P*  [a, b], then *m* ≤ *mr* ≤ *Mr* ≤ *M*.

or

or *m* (b-a) ≤ *L* (*P, ƒ*) ≤ *U* (*P, ƒ*) ≤ *M* (b-a)

In fact, if *P*1 , *P*2 **P**[a, b], then by Observation 5.

*M* (b-a) ≤ *L*(*P1 ,ƒ*) ≤ *U* (*P2 , ƒ*) ≤ *M* (b-a).

From this it follows, as in Observation 6.

But as ƒ *R*[a, b], we have

***Corollary 1:***  Denoting we get

***Corollary 2:***  If ƒ then there exists [a, b], such that

***Proof :*** Note ≤ *k* implies *k* ≤ ƒ ≤ *k.* Hence

We are now in position to prove.

***Theorem 1.2.2. :*** If ƒ, g *R*[a, b] and *c***R**, then

* cƒ
* ƒ±g *R*[a, b],
* ƒ.g [a, b],
* ƒ/g *R* [a, b] provided g>0 on [a, b],
* *R* [a, b],
* ƒ g *R* [a, b],
* g *R* [a, b].

***Proof :*** Before we can take up the proof of this theorem we need to prove the following lemma.

***Lemma* :** If ƒ: [a, b]→ **R** is bounded, then 0(ƒ)= sup {ƒ()} = *M*

Where 0(ƒ) denotes the oscillation of ƒ and *M*, *m* are respectively the supremum and infimum of ƒ over [a, b].

***Proof of Lemma :*** Clearly *m* ≤ ƒ(), ƒ() ≤ *M* for [a, b].

Hence ≤ *M* for [a, b]

i.e. *M* is an upper bound for

Next, let be arbitrary. Then as *M*  = sup ƒ and *m* = inf ƒ over [a, b], there exists 0, 0 [a, b] such that

ƒ(0) > *M* and ƒ(0) < *m*

Hence ƒ(0) 0) > *M m*

This implies *M m* is the supremum of when

i.e. *M m* = sup{, }.

***Proof of Theorem:*** *(i)* Since ƒ, for > 0 there exists *P*[a, b] such that

*W*(*P, ƒ*) <

Clearly boundedness of ƒ implies boundedness of *c*ƒ.

Now, let *Mr ’r* = sup where r

= sup where r

= sup {} = (*Mr r*)

Hence,

==

Therefore, *c*ƒ *R*[a, b]

(ii) Since ƒ,*g* *R*[a, b], for there exists *p* **P**[a, b] such that *w*(*P,* ƒ)< and *w*(*P, g*)< In fact, this *P* is the refinement of the partitions *P1 and P2*  obtained for ƒ and g respectively. Clearly ƒ is bounded.

Now, if *M*r, *mr* denote respectively the supremum and infimum of ƒg on *Ir*.

*M’*r, *m’*r denote respectively the supremum and infimum of ƒ on *Ir* and *M’r,m’r* denote respectively the supremum and infimum of g on *Ir, then*

*Mr mr* = sup {} where *Ir*

≤ sup {} sup {} where r

= *M’r* *m’r* *M’r* r

Hence

= *w*(*P,* ƒ) *w*(*P,* g)< =

Therefore, ƒg *R* [a, b]

Next, since g *R* [a, b] implies g *R* [a, b] by (ƒ),

ƒg = ƒ *R* [a, b].

(iii) We know ƒ, g[a, b] implies ƒ and g are bounded on [a, b].

Let *k, k’* + such that and g()≤*k’* over [a, b]. Further since ƒ, g [a, b] for , there

Exists *P* **P** [a, b] such that *w* (*P,* ƒ)<*k’* and *w* (*P,* g)< *k*.

Note, if *Mr, mr* denote te respectively the supremum and infimim of ƒ g on *Ir*.

*M’*r, *m’r* denote respectively the supremum and infimum of ƒ on *Ir*, and *M”r, m”r* denote respectively the supremum and infimum of g on *Ir*. Then we have

*Mr* *r* = sup {} where *Ir*

= sup {}

= sup {

≤ *k’* (*M’r* ) *k*(*M”r* *m”r*)

Hence,

=

Hence ƒg *R* [a, b]

(iv) Since ƒ, g *R* [a, b] and g> 0 on [a, b], we can find *k*, *k’*, *R*+ such that *k,* 0 on [a, b].

Also, for , there exists *P* such that *w*(*P,* ƒ) < 2/2*k* and *w* (*P, g*) < 2/2*k* . Clearly ƒ/g is bounded.

Now, if *Mr*, *mr* denote respectively the supremum and infimum of ƒ/g on *Ir*, *M’r*, *m’r* denote respectively the supremum and infimum of ƒ on *Ir*, *M”r,* *m”r* denote respectively the supremum and infimum of g on *Ir*, then

*Mr* r = sup where r

= sup where r

Hence, *w*(*P,* ƒ/g) =

= *k*’/

Hence ƒ/g *R* [a, b]

(v) Since ƒ *R* [a, b], for , there exists *P*  such that *w* (*P,* ƒ)< . Note boundedness of follows easily.

Let *Mr , mr* denote respectively the supremum and infimum of ƒ on *Ir*, *M’r, m’r*  denote respectively the supremum and infimum of on *Ir*.

Then

*M’r* m’r  = sup {} where r

< sup {} = *Mr mr*.

Hence,

Hence *R* [a, b]

(vi) Since, ƒ, g , ƒ±g [a, b] and therefore

[a, b],

We get (ƒ)(ƒ)

Hence, ƒ = max {ƒ, g} = ½ {(ƒ)} *R* [a, b]

(vii) Similarly ƒ, g implies ƒ±g

And

Hence ƒ = ½ {(ƒ)}

We now prove

To this end, we see

where *m’r* is the supremum of *c*ƒ on *Ip.*

Hence if *c* > 0

But if *c* > 0

Thus  
  
 Similarly, one can prove

Hence

Next, to we prove that

Since, ƒ,g, for , there exists such that for *P*  [a, b], < , *U*(*P,* ƒ) <

and *U* (*P,* g) <

[In fact, we get two different partitions of [a, b] for ƒ and g, but *P* is their common refinement.]

Also,

Since is arbitrary, we get

Had we procceded with instead of ƒ and g, we would have got

or

Combining we get

One can prove similarly

Finally we prove thatb

For this observe that ƒ and

Hence by Observation 8, we get

Hence,

**Remark 1:** ƒ, g implies ƒ = g the converse is not the true, e.g. Let

And

Then ƒ, g but ƒ .

**Remark 2:** ƒ, g implies ƒ, g but the converse is not true.

The example in Remark 1 above holds here also.

**Remark 3:** ƒ implies , but the converse is not true, e.g. Let

Then but ƒ.

**Remark 4:** If follows from the theorem that if ƒ , then ƒ3 . Note that the converse is not true. The example in Remark 3 holds here.

***Theorem 1.2.3.***  *(i)*  If ƒ and a < *c <* b, then

ƒ, ƒ

and conversely if ƒ *R* [a, c] and ƒ [c, b], then ƒ *R* [a, b].

Further

(ii) if ƒ [a, b] and [c, d], then ƒ

***Proof :*** (i) Leet ƒ for > 0, there exists *P* [a, b] such that *U* (*P,* ƒ)

Let *P\** = *P* . Then *U* (*P,* ƒ) ≥ *U* (*P\*,* ƒ) ≥ *L* (*P\*,* ƒ) ≥ *L* (*P,* ƒ)

Hence *U* (*P\*,* ƒ) ≤*U* (*P,* ƒ) .

Now, if *P1* =

and *P*2 =

then *P\** = *P*1 and

Clearly *U* (*P\*,* ƒ) = *U* (*P1,* ƒ),

*L* (*P\*,* ƒ) = *L* (*P1,* ƒ)

Hence [*U* (*P*1, ƒ)] = *U* (*P\*,* ƒ)

But since *U* (*P1,* ƒ)

*U* (*P1*, ƒ)

and *U* (*P2,* ƒ) *L* (*P2*, ƒ) <

This proves that ƒ and ƒ

Conversely, if ƒ and ƒ , then for there exists *P1*  and *P1* such that

*U* [*P1,* ƒ] and *U* (*P2,* ƒ) .

Now, if *P* = *P1* , then we get *P*  and for this *P*,

*U* (*P,* ƒ)

Hence ƒ

We now prove that  
  
 We have seen, if *P1*  **P**[a, c], *P1* , then

*U* (*P,* ƒ) = *U* (*P*1, ƒ)

Hence, inf *U* (*P,* ƒ) ≥ inf *U* (*P1*, ƒ) where *P*

i.e.

or

combining we get

or

since

(ii) By (i), ƒ

This completes the proof.

**1.3 INTEGRABLE FUNCTIONS**

In this section we shall investigate the types of functions that are integrable. We begin with continuous functions and see to what extent we can librate the conditions.

***Theorem 1.3.1 :*** Every continuous function over [a, b] is Riemann integrable.

***Proof :*** Let ƒ Then ƒ is bounded and uniformly continuous over [a, b].

Hence for there exists such that

whenever

Then *Mr* <

Let *P* and < Then

Hence ƒ *R* [a, b].

***Theorem 1.3.2 :*** A bounded function, continuous except at finitely many points of [a, b] is Riemann integrable on [a, b].

***Proof :***  Let ƒ be a bounded function, having discontinuous only at *p* points. We enclose these *p* points by very small closed intervals total length of which is less than (*M*) where *M* and *m*  are respectively the supremum and infimum of ƒ over [a, b]. If none of those points of discontinuity equals *a* and *b,* then we will be left with *p* closed subintervals on each of which ƒ is continuous. Therefore by the above theorem, we can find a partition *P1*, *i* = 1,2, ……, *p* of the closed subintervals such that

*U* (*Pi*, *t*)

Now if *P* = then

Let the second summation taken over the intervals covering the points of discontinuity

and since *Mr mr* ≤ *M*

=/2

*Hence* ƒ *R*[a, b].